JOURNAL OF APPROXIMATION THEORY 78, 30–40 (1994)

Summability of Hadamard Products of Taylor Sections with Polynomial Interpolants

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Communicated by T. J. Rivlin

Received August 3, 1991; accepted in revised form March 23, 1993

In previous papers the first author extended the classical equiconvergence theorem of Walsh by the application of summability methods in order to enlarge the disk of equiconvergence to regions of equisummability. The aim of this paper is to study equisummability of sequences which arise from Hadamard products of a fixed power series with Lagrange polynomial interpolants.

1. INTRODUCTION AND RESULTS

Let $g(z)=\sum_{\nu=0}^{\infty} g_\nu z^\nu$ be a power series with radius of convergence $R>1$, and let L_n^g be the Lagrange polynomial interpolating g in the $(n+1)$st roots of unity, i.e.,

$$L_n^g(\zeta_k)=g(\zeta_k) \qquad (k=0, 1, ..., n),$$

where $\zeta_k := \zeta_k^{(n)} := \exp(2\pi ik/(n+1))$. Denoting by S_n^g the nth partial sum of $\sum_{\nu=0}^{\infty} g_\nu z^\nu$, and setting $\mathbb{D}_\rho := \{z \in \mathbb{C}: |z|<\rho\}$, the classical equiconvergence theorem of Walsh [10, p. 153] may be stated as follows.

THEOREM. *For a power series $g(z)=\sum_{\nu=0}^{\infty} g_\nu z^\nu$ with radius of convergence $R>1$ we have*

$$\lim_{n\to\infty} [L_n^g(z)-S_n^g(z)]=0$$

compactly in $\mathbb{D}_{R^2}$ (i.e., uniformly on compact subsets of $\mathbb{D}_{R^2}$).

0021-9045/94 $6.00

In recent years many authors were concerned with various extensions of this theorem, where L_n^g was replaced by polynomials determined from a certain interpolation process, and S_n^g was replaced by polynomials determined from the power series of g. For a detailed survey we refer to the monographs of Sharma [8] and Varga [9, pp. 69–93].

In [3, 4] the first author applied certain summability methods for analytic continuation of power series in order to enlarge the disk of equiconvergence to regions of equisummability. The aim of this paper is to extend these results by studying the equisummability of the sequence

$$(L_n^g - S_n^g) * S_n^f = (L_n^g - S_n^g) * f,$$

where f is an arbitrary power series around the origin with positive radius of convergence. Thereby $*$ denotes the Hadamard product of two power series, i.e., if $F(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ and $G(z) = \sum_{\nu=0}^{\infty} b_\nu z^\nu$, then

$$(F * G)(z) := \sum_{\nu=0}^{\infty} a_\nu b_\nu z^\nu.$$

To state our results we introduce some notations.

Let $X \subset \mathbb{R}$, and let $x^* \in \mathbb{R} \cup \{\pm\infty\}$ be an accumulation point of X. Suppose further that $A = (a_n)_{n \in \mathbb{N}_0}$ is a sequence of functions on X. A sequence $(s_n)_{n \in \mathbb{N}_0}$ of functions defined in an open set $\Omega \subset \mathbb{C}$ is called compactly A-summable in Ω to the function s, if

$$\sigma(x, z) := \sum_{n=0}^{\infty} a_n(x)\, s_n(z) \qquad (z \in \Omega,\ x \in X)$$

converges compactly in Ω for all $x \in X$, and if

$$\lim_{x \to x^*} \sigma(x, z) = s(z)$$

compactly in Ω. In this case we use the abbreviation

$$A\text{-}\lim_{n \to \infty} s_n(z) = s(z) \qquad \text{compactly in } \Omega.$$

The geometric series $\sum_{\nu=0}^{\infty} z^\nu$ as well as its analytic continuation $1/(1-z)$ will be denoted by $\gamma(z)$. Finally, for arbitrary sets A, $B \subset \mathbb{C}$ and $\alpha \in \mathbb{C}$, $k \in \mathbb{N}$ we define

$$A^c := \mathbb{C} \backslash A, \qquad A \cdot B := \{ab\colon a \in A, b \in B\}, \qquad \alpha A := \{\alpha a\colon a \in A\},$$

$$A^k := \{a^k\colon a \in A\}, \qquad A_{(k)} := \left(\bigcap_{j=0}^{k-1} e^{2\pi i j/k} A\right)^k,$$

and

$$A * B := (A^c \cdot B^c)^c.$$

Now we can state our main theorem as follows.

THEOREM 1. *Let $\Omega \subset \mathbb{C}$ be an open set with $0 \in \Omega$, and let f be holomorphic in Ω with $f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu$ around the origin. Suppose further that $A = (a_n)$ is such that the following condition $(*)$ holds.*

$(*)$ *With $\rho := \sup_{z \in \Omega} |z|$, the power series in two variables*

$$\phi^f(x; z, w) := \sum_{n, \nu = 0}^{\infty} a_{n+\nu}(x) f_\nu z^\nu w^n$$

converges for all $x \in X$ and $(z, w) \in \mathbb{D}_\rho \times \mathbb{D}$, and

$$\lim_{x \to x^*} \phi^f(x; z, w) = 0 \qquad \text{compactly in } (z, w) \in \Omega \times \mathbb{D}.$$

If $g(z) = \sum_{\nu=0}^{\infty} g_\nu z^\nu$ defines a function holomorphic in a region $G \subset \mathbb{C}$ with $\mathbb{D}_R \subset G$ for some $R > 1$, then

$$A\text{-}\lim_{n \to \infty} ((L_n^g - S_n^g) * S_n^f)(z) = 0 \qquad \text{compactly in } \mathscr{E} := \bigcap_{k \geqslant 2} G_{(k)} * \Omega.$$

Writing $(L_n^g * S_n^f)(z) = ((L_n^g - S_n^g) * S_n^f)(z) + S_n^{g*f}(z)$, a combination of Theorem 1 and [6, Corollary 2] leads readily to

THEOREM 2. *Let Ω, f, A, and g be as in Theorem 1. Suppose in addition that f is holomorphic in an open set $\Omega' \subset \mathbb{C}$ with $\Omega \subset \Omega'$, and that*

$$A\text{-}\lim_{n \to \infty} S_n^f(z) = f(z) \qquad \text{compactly in } \Omega.$$

Then there exists a function $g * f$ holomorphic in $G * \Omega'$ such that

$$(g * f)(z) = \sum_{\nu=0}^{\infty} g_\nu f_\nu z^\nu$$

in a neighborhood of the origin, and we have

$$A\text{-}\lim_{n \to \infty} (L_n^g * S_n^f)(z) = (g * f)(z) \qquad \text{compactly in } \mathscr{L} := \bigcap_{k \geqslant 1} G_{(k)} * \Omega.$$

Hence, $A\text{-}\lim_{n \to \infty} (L_n^g * S_n^f)(z)$ represents an analytic continuation of $\sum_{\nu=0}^{\infty} g_\nu f_\nu z^\nu$ in $\mathscr{L} \cap (G * \Omega')_0$, where $(G * \Omega')_0$ denotes the component of $G * \Omega'$ which contains the origin.

The crucial assumption in Theorem 1 and 2 is the condition $(*)$. The following lemma gives a sufficient condition, under which $(*)$ is satisfied.

LEMMA. *Let f be a holomorphic in an open set $\Omega' \subset \mathbb{C}$ with $\mathbb{D} \subset \Omega'$, and let $A = (a_n)$ be such that*

$$A\text{-}\lim_{n \to \infty} S_n^\gamma(z) = \gamma(z)$$

compactly in an open set $S \subset \mathbb{C} \setminus \{1\}$ with $\Omega \subset \Omega' * S$ and $\mathbb{D} \subset S$. Then condition $(*)$ is satisfied, and

$$A\text{-}\lim_{n \to \infty} S_n^f(z) = f(z) \qquad \textit{compactly in } \Omega.$$

Remark 1. If f is holomorphic in $\Omega' = \mathbb{C} \setminus \{1\}$, then $\Omega' * \Omega = \Omega$ for arbitrary $\Omega \subset \mathbb{C}$, and thus every matrix $A = (a_n)$ with

$$A\text{-}\lim_{n \to \infty} S_n^\gamma(z) = \gamma(z) \qquad \text{compactly in } \Omega$$

satisfies condition $(*)$ and

$$A\text{-}\lim_{n \to \infty} S_n^f(z) = f(z) \qquad \text{compactly in } \Omega.$$

This is also true if f is holomorphic in $\mathbb{C} \setminus [1, \infty)$, and if Ω is starlike (with respect to the origin). For the classical methods of analytic continuation by summability (Euler's methods, Borel's method, Lindelöf's method, etc.) the region Ω of summability of γ is starlike. In Theorem 2 we have in this case $\mathscr{L} \cap (G * \Omega')_0 = \mathscr{L}$ since $(G * \Omega')_0 = G * \Omega' \supset G * \Omega$.

Remarks 2. (1) Combining Theorem 1 and Theorem 2, respectively, with the lemma in the special case $f = \gamma$, $\Omega' = \mathbb{C} \setminus \{1\}$, and $S = \Omega$, we get Theorem 1 and Theorem 2 in [4], respectively. In general, $\mathscr{E}$ and $\mathscr{L}$ are open sets (but not necessarily connected), and if $\mathbb{D}_{R'} \subset \Omega$ for some $R' > 0$, then $\mathscr{E} \supset \mathbb{D}_{R'R^2}$ and $\mathscr{L} \supset \mathbb{D}_{R'R}$. A short calculation shows that $G_{(k)} * \Omega = \bigcap_{c \notin G} c^k \Omega$. If one of the sets G or Ω is starlike (with respect to the origin), then $\mathscr{E}$ and $\mathscr{L}$ are also starlike. Further, if $\Omega = \mathbb{C} \setminus [1, \infty)$, and if G is starlike, then $G_{(k)} * \Omega = G_{(k)}$.

(2) Under the assumptions of Theorem 1 the Cauchy integral formula gives for any $r \in (1, R)$

$$S_n^{g * f}(z) = \frac{1}{2\pi i} \int_{|\zeta| = r} g(\zeta)\, S_n^f\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta} \qquad (z \in \mathbb{C}), \tag{1}$$

and by the well-known Hermite interpolation formula (see, for example, [5, pp. 61–62] or [10, p. 50]), L_n^g may be written as

$$L_n^g(z)=\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{g(\zeta)}{\zeta-z}\,\frac{\zeta^{n+1}-z^{n+1}}{\zeta^{n+1}-1}\,d\zeta \qquad (z\in\mathbb{C}),$$

and thus

$$(L_n^g * S_n^f)(z):=\frac{1}{2\pi i}\int_{|\zeta|=r} g(\zeta)\,S_n^f\!\left(\frac{z}{\zeta}\right)\frac{\zeta^{n+1}}{\zeta^{n+1}-1}\,\frac{d\zeta}{\zeta} \qquad (z\in\mathbb{C}). \tag{2}$$

The residue theorem shows that

$$(L_n^g * S_n^f)(z)=\frac{1}{n+1}\sum_{k=0}^{n} g(\zeta_k)\,S_n^f\!\left(\frac{z}{\zeta_k}\right),$$

where the $\zeta_k=\exp(2\pi ik/(n+1))$ are the $(n+1)$st roots of unity. It could be of interest to remark that $L_n^g * S_n^f$ may be viewed as a discretization of the integral (1) representing S_n^{g*f}.

(3) Setting $g(z):=\gamma(R^{-1}z)$ and $G:=\mathbb{C}\setminus\{R\}$ in Theorem 1 and Theorem 2, respectively, we obtain an equiconvergence result for the difference

$$(L_n^g * S_n^f)(z)-S_n^f\!\left(\frac{z}{R}\right)=\left[\frac{1}{n+1}\sum_{k=0}^{n}\frac{R}{R-\zeta_k}\,S_n^f\!\left(\frac{z}{\zeta_k}\right)\right]-S_n^f\!\left(\frac{z}{R}\right),$$

namely that for $A=(a_n)$ such that condition $(*)$ holds,

$$A\text{-}\lim_{n\to\infty}\left[(L_n^g * S_n^f)(z)-S_n^f\!\left(\frac{z}{R}\right)\right]=0 \qquad \text{compactly in } \bigcap_{k\geqslant 2} R^k\Omega$$

and

$$A\text{-}\lim_{n\to\infty}(L_n^g * S_n^f)(z)=f\!\left(\frac{z}{R}\right) \qquad \text{compactly in } \bigcap_{k\geqslant 1} R^k\Omega.$$

For a power series $f(z)=\sum_{\nu=0}^{\infty} f_\nu z^\nu$ we set formally

$$f_{-1}(z):=\sum_{\nu:f_\nu\neq 0} f_\nu^{-1}z^\nu.$$

Replacing $g(z)=\sum_{\nu=0}^{\infty} g_\nu z^\nu$ by $h(z):=(g*f_{-1})(z)=\sum_{\nu:f_\nu\neq 0} g_\nu f_\nu^{-1}z^\nu$ in the assumptions of Theorems 1 and 2, we have in the case that $g_\nu=0$ for all ν such that $f_\nu=0$ the identity $S_n^{h*f}=S_n^g$. Therefore, we find the following result concerning the equisummability of $L_n^h * S_n^f$ and the partial sums S_n^g of g.

COROLLARY. *Let Ω, f, and A be as in Theorem 2. If* $h(z) := \sum_{\nu : f_\nu \neq 0} g_\nu f_\nu^{-1} z^\nu$ *defines a function holomorphic in a region* $G \subset \mathbb{C}$ *with* $\mathbb{D}_R \subset G$ *for some* $R > 1$, *then*

$$A\text{-}\lim_{n \to \infty} [(L_n^h * S_n^f)(z) - S_n^g(z)] = 0 \qquad \textit{compactly in } \mathscr{E}$$

and

$$A\text{-}\lim_{n \to \infty} (L_n^h * S_n^f)(z) = g(z) \qquad \textit{compactly in} \quad \mathscr{L} \cap (G * \Omega')_0,$$

where $\mathscr{E}$ and $\mathscr{L}$ are defined as in Theorems 1 *and* 2.

EXAMPLE. Let ϕ be an entire function of exponential type zero such that $\phi(\nu) \neq 0$ for all $\nu \in \mathbb{N}_0$ and $\lim_{\nu \to \infty} |\phi(\nu)|^{1/\nu} = 1$. If, in addition, ϕ has only finitely many zeros in the angle $\{w \in \mathbb{C}\colon -\alpha \leqslant \arg w \leqslant \beta\}$ for some $\alpha, \beta \in (0, \pi/2)$, then by a result of Agmon (see [1] or [2, pp. 145 and 153]), the power series

$$f(z) := \sum_{\nu=0}^{\infty} \frac{z^\nu}{\phi(\nu)} \qquad (z \in \mathbb{D})$$

defines a function holomorphic in the (starlike) domain $\Omega_{\alpha, \beta}$ bounded by the logarithmic spirals

$$\{e^{\vartheta(\tan \alpha + i)} : 0 \leqslant \vartheta \leqslant \vartheta_0\} \qquad \text{and} \qquad \{e^{(2\pi - \vartheta) \tan \beta + i\vartheta} : \vartheta_0 \leqslant \vartheta \leqslant 2\pi\},$$

where $\vartheta_0 := 2\pi \tan \beta/(\tan \alpha + \tan \beta)$. In particular, if ϕ has no zeros in $\operatorname{Re} w > 0$, then f is holomorphic in $\mathbb{C} \setminus [1, \infty)$. By the well-known theorem of Wigert (see, for example, [2, p. 8]), the power series

$$f_{-1}(z) = \sum_{\nu=0}^{\infty} \phi(\nu)\, z^\nu$$

defines a function holomorphic in $\hat{\mathbb{C}} \setminus \{1\}$. Thus, by the Hadamard multiplication theorem (cf. [6, Theorem H]), for an arbitrary function g holomorphic in a region $G = G(g)$ with $0 \in G$, the function

$$h(z) := (g * f_{-1})(z) = \sum_{\nu=0}^{\infty} g_\nu \phi(\nu)\, z^\nu$$

is also holomorphic in G. (The sequence $(\phi(\nu))_{\nu \in \mathbb{N}_0}$ is a so-called holomorphy preserving factor sequence.) However, if

$$g(z) := \sum_{\nu=0}^{\infty} \frac{z^\nu}{\phi_1(\nu)}$$

with an entire function ϕ_1 such that $\phi_0 = \phi/\phi_1$ is also an entire function of exponential type zero, then $h(z) = \sum_{\nu=0}^{\infty} \phi_0(\nu) z^\nu$ is holomorphic in $G(h) = \mathbb{C} \setminus \{1\}$. Therefore, we always have $G(g) \subset G(h)$, and in certain cases $G(g) \subsetneqq G(h)$. Comparing the assertions of Theorems 1 and 2 in [4] with the assertion in the corollary we find that the equisummability results for $(L_n^h * S_n^f - S_n^g)$ are stronger than the results for the "classical" difference $(L_n^g - S_n^g)$.

2. Proofs

We now turn to the proofs of Theorem 1 and the lemma.

Proof of Theorem 1. By (1) and (2) we have for any $r \in (1, R)$

$$\begin{aligned}
((L_n^g - S_n^g) * S_n^f)(z) &= \frac{1}{2\pi i} \int_{|\zeta|=r} S_n^f\left(\frac{z}{\zeta}\right) \frac{1}{\zeta^{n+1}-1} g(\zeta) \frac{d\zeta}{\zeta} \\
&= \frac{1}{2\pi i} \int_{|\zeta|=r} \left(\sum_{\nu=0}^{n} f_\nu \left(\frac{z}{\zeta}\right)^\nu\right) \left(\sum_{k=0}^{\infty} \frac{1}{\zeta^{(k+1)(n+1)}}\right) g(\zeta) \frac{d\zeta}{\zeta} \\
&= \frac{1}{2\pi i} \int_{|\zeta|=r} \sum_{k=0}^{\infty} \sum_{\nu=0}^{n} f_\nu \left(\frac{z}{\zeta}\right)^\nu \frac{1}{\zeta^{(k+1)(n+1)}} g(\zeta) \frac{d\zeta}{\zeta}.
\end{aligned}$$

Applying the matrix method $A = (a_n)$ to this difference, we obtain for $|z| < \rho r^2$

$$\begin{aligned}
\psi(x, z) &:= \sum_{n=0}^{\infty} a_n(x)((L_n^g - S_n^g) * S_n^f)(z) \\
&= \frac{1}{2\pi i} \int_{|\zeta|=r} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\nu=0}^{n} a_n(x) f_\nu \left(\frac{z}{\zeta}\right)^\nu \frac{1}{\zeta^{(k+1)(n+1)}} g(\zeta) \frac{d\zeta}{\zeta} \\
&= \frac{1}{2\pi i} \int_{|\zeta|=r} \sum_{k=0}^{\infty} \sum_{\nu=0}^{\infty} \sum_{n=\nu}^{\infty} a_n(x) f_\nu \left(\frac{z}{\zeta}\right)^\nu \frac{1}{\zeta^{(k+1)(n+1)}} g(\zeta) \frac{d\zeta}{\zeta} \\
&= \frac{1}{2\pi i} \int_{|\zeta|=r} \sum_{k=0}^{\infty} \sum_{\nu=0}^{\infty} \sum_{n=0}^{\infty} a_{n+\nu}(x) f_\nu \left(\frac{z}{\zeta}\right)^\nu \frac{1}{\zeta^{(k+1)(n+\nu+1)}} g(\zeta) \frac{d\zeta}{\zeta} \\
&= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{1}{\zeta^{k+1}} \phi^f\left(x; \frac{z}{\zeta^{k+2}}, \frac{1}{\zeta^{k+1}}\right) g(\zeta) \frac{d\zeta}{\zeta}.
\end{aligned}$$

The changes of summation and integration in the above computation are justified by the first part of condition $(*)$. In addition, $\psi(x, \cdot)$ is a holomorphic function in $\mathbb{D}_{\rho R^2}$, since r can be chosen arbitrarily close to R.

Now consider a compact subset C of $\mathscr{E}$. Without loss of generality we may assume that $\bar{\mathbb{D}}_{r^2} \subset C$. Let $\mu := \max_{z \in C} |z|$, and choose $N \in \mathbb{N}$ such that

$$\frac{\mu}{r^{N+2}} \leqslant \delta,$$

where $\bar{\mathbb{D}}_\delta \subset \Omega$. Then we have

$$\left|\frac{z}{\zeta^{k+2}}\right| \leqslant \delta \qquad \text{for} \quad |\zeta| = r, \ |z| \leqslant \mu, \text{ and } k \geqslant N.$$

We split $\psi(x, z)$ into two sums

$$\psi(x, z) = \psi_1(x, z) + \psi_2(x, z),$$

where the first sum ranges over all k with $0 \leqslant k \leqslant N-1$, and the second sum ranges over all $k \geqslant N$.

For ψ_2 we have

$$|\psi_2(x, z)| \leqslant \sum_{k=N}^{\infty} \frac{1}{2\pi} \int_{|\zeta| = r} \frac{1}{r^{k+1}} \left|\phi^f\left(x; \frac{z}{\zeta^{k+2}}, \frac{1}{\zeta^{k+1}}\right)\right| |g(\zeta)| \frac{|d\zeta|}{r}.$$

Setting $M(r, g) := \max_{|\zeta| = r} |g(\zeta)|$ and

$$\varepsilon(x) := \max_{\substack{|w| \leqslant 1/r \\ |u| \leqslant \delta}} |\phi^f(x; u, w)|,$$

we get by condition $(*)$

$$\max_{z \in C} |\psi_2(x, z)| \leqslant \varepsilon(x)\, M(r, g) \sum_{k=N}^{\infty} \frac{1}{r^{k+1}} \leqslant \varepsilon(x) \frac{M(r, g)}{r-1} \to 0 \qquad (x \to x^*). \tag{3}$$

For the estimate of ψ_1 it suffices to consider a single term

$$\frac{1}{2\pi i} \int_{|\zeta| = r} \frac{1}{\zeta^{k+1}} \phi^f\left(x; \frac{z}{\zeta^{k+2}}, \frac{1}{\zeta^{k+1}}\right) g(\zeta) \frac{d\zeta}{\zeta} \tag{4}$$

for fixed $0 \leqslant k \leqslant N-1$.

We first make some topological preliminaries. A short calculation shows that

$$G_{(k+2)} * \Omega = \{z \in \mathbb{C} : zE \subset G_{(k+2)}\},$$

where $E := \{\zeta^{-1} : \zeta \notin \Omega\} \cup \{0\}$. Since C is a compact subset of $G_{(k+2)} * \Omega$, and E is compact in $\mathbb{C}$, the set $C \cdot E$ is a compact subset of $G_{(k+2)}$. It follows that $K := \{\omega \in \mathbb{C} : \omega^{k+2} \in C \cdot E\}$ is a compact subset of G. From standard topology it is known that there exists a cycle Γ in $G \backslash (K \cup \bar{\mathbb{D}}_r)$ such that

$$\operatorname{ind}_\Gamma(\omega) = \begin{cases} 1, & \omega \in K \cup \bar{\mathbb{D}}_r \\ 0, & \omega \notin G \end{cases}$$

(cf. [7, pp. 268–269]). Therefore, we can replace the path of integration in (4) by Γ. From the construction of Γ it follows that

$$B := \{z/\zeta^{k+2} : z \in C, \zeta \in \Gamma\}$$

is a compact subset of Ω.

Setting

$$\varepsilon_1(x) := \max_{\substack{|w| \leqslant 1/r \\ u \in B}} |\phi^f(x; u, w)|,$$

we obtain by condition (*)

$$\max_{z \in C} \left| \frac{1}{2\pi i} \int_\Gamma \frac{1}{\zeta^{k+1}} \phi^f\left(x; \frac{z}{\zeta^{k+2}}, \frac{1}{\zeta^{k+1}}\right) g(\zeta) \frac{d\zeta}{\zeta} \right|$$
$$\leqslant \frac{1}{2\pi} \max_{\zeta \in \Gamma} |g(\zeta)| \frac{\operatorname{length}(\Gamma)}{r^{k+2}} \varepsilon_1(x) \to 0 \qquad (x \to x^*). \tag{5}$$

From (3), (4), and (5) it follows that

$$\max_{z \in C} |\psi(x, z)| \to 0 \qquad (x \to x^*),$$

and this completes the proof. ∎

Proof of the Lemma. Let $A = (a_n)$ be such that for an open set $S \subset \mathbb{C} \backslash \{1\}$ with $\mathbb{D} \subset S$ and $\Omega' * S \supset \Omega$

$$A\text{-}\lim_{n \to \infty} S_n^\gamma(z) = \gamma(z) \qquad \text{compactly in } S.$$

Since $\sum_{n=0}^\infty a_n(x) = \sum_{n=0}^\infty a_n(x)\, S_n^\gamma(0) \to 1$ $(x \to x^*)$, the power series

$$z \sum_{n=0}^\infty a_n(x)\, z^n = (z-1) \left[\sum_{n=0}^\infty a_n(x)\, S_n^\gamma(z) \right] + \sum_{n=0}^\infty a_n(x)$$

has radius of convergence at least $\rho_S := \sup_{z \in S} |z| \in [1, \infty]$, and we obtain

$$\lim_{x \to x^*} \sum_{n=0}^\infty a_n(x)\, z^n = 0 \qquad \text{compactly in } S. \tag{6}$$

It is easy to verify that for $x \in X$ the power series in two variables

$$\phi^\gamma(x; z, w) = \sum_{n, \nu = 0}^{\infty} a_{n+\nu}(x)\, z^\nu w^n$$

converges in $\mathbb{D}^2_{\rho S}$. We compute for $w \neq 0$

$$\begin{aligned}
\phi^\gamma(x; z, w) &= \sum_{n=0}^{\infty} w^n \sum_{\nu=0}^{\infty} a_{n+\nu}(x)\, z^\nu = \sum_{n=0}^{\infty} w^n \sum_{\nu=n}^{\infty} a_\nu(x)\, z^{\nu-n} \\
&= \sum_{\nu=0}^{\infty} a_\nu(x) \sum_{n=0}^{\nu} w^n z^{\nu-n} = \sum_{\nu=0}^{\infty} a_\nu(x)\, w^\nu \frac{1-(z/w)^{\nu+1}}{1-z/w} \\
&= \frac{1}{1-z/w} \sum_{\nu=0}^{\infty} a_\nu(x)\, w^\nu - \frac{z/w}{1-z/w} \sum_{\nu=0}^{\infty} a_\nu(x)\, z^\nu, \qquad (7)
\end{aligned}$$

and for $w = 0$

$$\phi^\gamma(x; z, 0) = \sum_{\nu=0}^{\infty} a_\nu(x)\, z^\nu. \qquad (8)$$

Let $r < 1$, and let K be a compact subset of S. Without loss of generality we may assume that $\bar{\mathbb{D}}_{r_0} \subset K$ for some $r_0 \in (r, 1)$. For $0 < |w| \leqslant r$ and $z \in K$, $|z| \geqslant r_0$ we have

$$\left|\frac{1}{1-z/w}\right| = \left|\frac{w}{w-z}\right| \leqslant \frac{r}{r_0 - r},$$

and

$$\left|\frac{z/w}{1-z/w}\right| = \left|\frac{z}{w-z}\right| \leqslant \frac{1}{r_0 - r} \max_{z \in K} |z|.$$

By (6), (7), and (8) we find

$$\lim_{x \to x^*} \phi^\gamma(x; z, w) = 0 \qquad \text{uniformly in} \quad |w| \leqslant r \text{ and } z \in K,\ |z| \geqslant r_0,$$

and by the maximum principle

$$\lim_{x \to x^*} \phi^\gamma(x; z, w) = 0 \qquad \text{compactly in} \quad (z, w) \in S \times \mathbb{D}.$$

Let $H(U)$ be the topological vector space of all functions holomorphic in the open set $U \subset \mathbb{C}$ with the usual topology of compact convergence. Then the linear mapping $T_f : H(S) \to H(\Omega' * S)$ defined by

$$T_f(\varphi) := f * \varphi \qquad (\varphi \in H(S))$$

is continuous (cf. [6]). The above considerations show that for $r < 1$ the family

$$\{\phi^{\gamma}(x; \cdot, w) : |w| \leqslant r\}$$

is uniformly convergent in $H(S)$ to the limit function zero as x tends to x^*. Hence

$$\lim_{x \to x^*} \phi^{f}(x; \cdot, w) = \lim_{x \to x^*} f * \phi^{\gamma}(x; \cdot, w) = 0$$

in $H(\Omega' * S)$ uniformly for $|w| \leqslant r$. Since $\Omega \subset \Omega' * S$, we conclude that

$$\lim_{x \to x^*} \phi^{f}(x; z, w) = 0 \qquad \text{compactly in} \quad (z, w) \in \Omega \times \mathbb{D}.$$

The first part of condition $(*)$ is necessarily satisfied since $\phi^{f}(x; z, w)$ is a power series in (z, w). The last assertion follows from Corollary 2 in [6]. ∎

Acknowledgment

The authors thank the referee for several helpful suggestions which have been incorporated into the text.

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